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Wigner Function: from ``quasi-particles” to stochastic radiation patterns

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Outline

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- Particle/wave mechanics and phase-plane
- Wigner function: properties
- Radiation from complex sources
- Role of correlation function
- Towards *stochastic* radiation patterns
- Results
- Conclusions

Parabolic wave equation

Reduced function for principal direction z

$$\varphi(x,z) = e^{ikz}\psi(x,z)$$

Scalar wave equation for inhomogeneous medium

$$\frac{\partial^2\varphi}{\partial z^2} + 2ik\frac{\partial\varphi}{\partial z} + \frac{\partial^2\varphi}{\partial x^2} + k^2\left(\varepsilon(x)-1\right)\varphi = 0$$

Formally factored as $\left\{\frac{\partial}{\partial z} - \left(\hat{Q}-i\right)\right\}\left\{\frac{\partial}{\partial z} - \left(-\hat{Q}-i\right)\right\}\varphi = 0$

With pseudo-differential operator

$$\hat{Q} = -i\sqrt{k^2\varepsilon(x) - \frac{\partial^2}{\partial x^2}}$$

$+\hat{Q}$ **Regressive wave**

$-\hat{Q}$ **Progressive wave**

M. Levy; ``Parabolic Equation Methods for Electromagnetic Wave Propagation'', *IET* 2000.

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Waves in phase-plane

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<<Wave propagation in overmoded structures may be described via a […] phase-(plane) ``kinematic" equation derivable exactly from the wave equation>>

Nathan Marcuvitz

Time domain parabolic $\frac{\partial \varphi}{\partial t} = -i\sqrt{\Omega^2(x) - c^2 \frac{\partial^2}{\partial x^2}}\varphi \longrightarrow \varphi(x,t)$

From a knowledge of $\varphi(x,0)$

Form density function $\rho_t(x_1,x_2) = \varphi(x_1,t)\varphi^*(x_2,t) \longrightarrow x = \frac{x_1+x_2}{2}$, $s = x_1 - x_2$

Take Fourier (Wigner) transform $W_t(x,k) = \int_{-\infty}^{\infty} \rho_t\left(x+\frac{s}{2}, x-\frac{s}{2}\right)\exp(-iks)\,ds$

N. Marcuvitz; Proceedings of the IEEE. **79** 10, October 1991

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Connection with (quantum) wavefunctions

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$$\frac{\partial\varphi}{\partial t} = -i\sqrt{\Omega^2(x) - c^2\frac{\partial^2}{\partial x^2}}\varphi \longrightarrow -\left(\frac{\hbar}{i}\right)\frac{\partial\varphi}{\partial t} = \sqrt{\eta(x) - \hbar^2 c^2\frac{\partial^2}{\partial x^2}}\varphi$$

$$\eta(x) = \frac{\Omega^2(x)}{\Omega_0^2}$$

$$\hbar = \frac{1}{\Omega_0}$$

High frequency asymptotics $\hbar \rightarrow 0$ and weak inhomogeneity $\eta(x) \cong 1$ yield

$$-\left(\frac{\hbar}{i}\right)\frac{\partial\varphi}{\partial t} = \left[\frac{1}{2}\left(\frac{\hbar c}{i}\frac{\partial}{\partial x}\right)^2 + V(x)\right]\varphi \qquad V(x) = \frac{1}{2}\left[\eta(x) - 1\right]$$

Schroedinger equation

N. Marcuvitz; Proceedings of the IEEE. **79** 10, October 1991

Wave ``Hamiltonian''

z-dependent problem $t \to -z$

$$\frac{\partial \varphi}{\partial z} = -i\sqrt{k_0^2 \varepsilon(x) - \frac{\partial^2}{\partial x^2}}\varphi \longrightarrow \left(\frac{\hbar}{i}\right)\frac{\partial \varphi}{\partial z} = \sqrt{\varepsilon(x) - \hbar^2 \frac{\partial^2}{\partial x^2}}\varphi \qquad \hbar = \frac{1}{k_0}$$

Complications:

- singular nature of square root operator
- *x* does not commute with $\frac{\hbar}{i}\frac{\partial}{\partial x}$

$$\hat{H} = \sqrt{\varepsilon(x) - \frac{\partial^2}{\partial x^2}}$$

$$\uparrow$$

Cures:

- formal solution in terms of operators $\varphi(x,z) = \exp\left[\left(\frac{i}{\hbar}\right)\hat{H}z\right]\varphi(x,0)$
- Iterative numerical computation $\varphi(x,z+\Delta z) \leftarrow \varphi(x,z)$

N. Marcuvitz; Proceedings of the IEEE. **79** 10, October 1991

Matrix representation

Green operator $\hat{G}$ needs an explicit representation $\hat{G} \to G(x_1, x_2)$

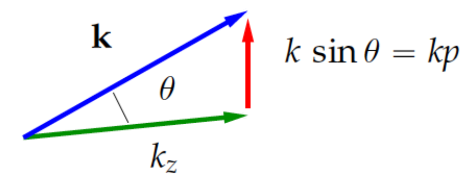
$$\hat{G} = \exp\left[\left(\frac{i}{\hbar}\right)\hat{H}z\right] \to \varphi(x,z) = \int G\left(x,x^{'},z\right)\varphi(x,0)\,dx^{'}$$

Procedure*:

1. Find matrix representation of Hamiltonian $\hat{H} \to H(x_1, x_2)$
2. Calculate Weyl transform of $H(x_1,x_2) \to h(p,x) \cong \sqrt{\varepsilon(x) - p^2}$
3. Find matrix representation of Green's function

$$G(x_1,x_2;z) \cong \frac{1}{2\pi\hbar}\int \exp\left[\left(\frac{i}{\hbar}\right)h(p,x_1)z + p(x_1 - x_2)\right]dp$$

$$k = \frac{p}{\hbar}$$



* Exercise: prove matrix representation following *Appendix* of N. Marcuvitz; Proceedings of the IEEE. **79** 10, October 1991

Evolution of time-domain density operator

Take the time derivative of the density operator

$$\frac{\partial \rho_t(x_1,x_2)}{\partial t} = \frac{\partial}{\partial t}\left(\varphi(x_1,t)\varphi^*(x_2,t)\right)$$

Define commutator

$$\left[\hat{H},\rho_t\right] = \hat{H}\rho_t - \rho_t\hat{H}$$

Assumes the form of a ``kinetic'' equation

$$\frac{\partial \rho_t}{\partial t} = -\frac{i}{\hbar}\left[\hat{H},\rho_t\right] = -\frac{i}{\hbar}\hat{L}\rho_t$$

With formal iterative solution

$$\rho_{t+\Delta t} = \exp\left(-\frac{i}{\hbar}\hat{L}\Delta t\right)\rho_t$$

N. Marcuvitz; Proceedings of the IEEE. **79** 10, October 1991

``Collisionless'' kinetic equation

Take the Wigner transform of

$$\frac{\partial W_t}{\partial t} = -\frac{i}{\hbar}\left[h, W_t\right] = -\frac{i}{\hbar}\hat{l}W_t$$

with dimensionless operator in phase-plane*

$$\hat{l} = \frac{2}{i}\sin\left[\left(\frac{\hbar}{2}\right)\left(\frac{\partial^h}{\partial p}\frac{\partial}{\partial x} - \frac{\partial^h}{\partial x}\frac{\partial}{\partial p}\right)\right]h\left(p,x\right)$$

Zero-order expansion of $\hat{l}$ in $\hbar$ yields

$$\frac{\partial W_t}{\partial t} = \left(\frac{\partial h}{\partial x}\frac{\partial}{\partial p} - \frac{\partial h}{\partial p}\frac{\partial}{\partial x}\right)W_t \rightarrow W_{t+\Delta t}\left(p,x\right) = W_t\left(p + \Delta t\frac{\partial h}{\partial x}, x - \Delta t\frac{\partial h}{\partial p}\right)$$

Quasiparticle centroid shears!

* Exercise: prove phase-space operator following *Appendix* of N. Marcuvitz; Proceedings of the IEEE. **79** 10, October 1991

Alternative derivation (1)

From $\frac{\partial W_t}{\partial t} = -\frac{i}{\hbar}\hat{l}W_t$ find a Green's operator $\hat{G} = \exp\left[-\left(\frac{i}{\hbar}\right)\hat{l}\,\Delta t\right]$

and its matrix representation

$$G\left(p,x;p',x';\Delta t\right) \cong \frac{1}{4\pi^2}\iint \exp\left(-\frac{i}{\hbar}\hat{l}\,\Delta t\right)\exp\left[\kappa\left(x-x'\right)-\chi\left(p-p'\right)\right]d\kappa\,d\chi$$

phase-plane dynamics is obtained

$$W_{t+\Delta t}\left(p,x\right) = \iint G\left(p,x;p',x';\Delta t\right)W_t\left(p',x'\right)dp'\,dx'$$

Zero-order expansion of $\hat{l}$ in $\hbar$ yields

$$W_{t+\Delta t}\left(p,x\right) = W_t\left(p+\Delta t\frac{\partial h}{\partial x}, x-\Delta t\frac{\partial h}{\partial p}\right)$$

N. Marcuvitz; Proceedings of the IEEE. **79** 10, October 1991

Alternative derivation (2)

1. Take derivative of Wigner function definition w.r.t. transport variable;

$$\frac{\partial W_z(x,p)}{\partial z} = \left(\frac{k}{2\pi}\right)^d \int_{-\infty}^{\infty} \frac{\partial \varphi_z\left(x+\frac{s}{2}\right)}{\partial z} \varphi_z^*\left(x-\frac{s}{2}\right) + \varphi_z\left(x+\frac{s}{2}\right) \frac{\partial \varphi_z^*\left(x-\frac{s}{2}\right)}{\partial z} \exp(-ikps)\, ds$$

2. Use ``small-angle'' Helmholtz equation $\frac{\partial \varphi_z}{\partial z} = i\sqrt{k^2 + \frac{\partial^2}{\partial x^2}}\varphi_z$

3. Integrate by parts

4. Obtain transport equation $\frac{\partial W_z(x,p)}{\partial z} = -\frac{\sqrt{1-p^2}}{p}\frac{\partial W_z(x,p)}{\partial x}$

5. Solve

$$W_z(x,p) = W_{z=0}\left(x - z\frac{p}{\sqrt{1-p^2}}, p\right)$$

Question: **free-space?**

$$h(p,x) \cong \sqrt{\varepsilon(x) - p^2}$$

$$\hbar \to 0 \quad \varepsilon(x) = 1$$

Casey ``Wigner function for pedestrians'', Am. J. of Phys. (2008) ; Besieris & Tappert, 90s.

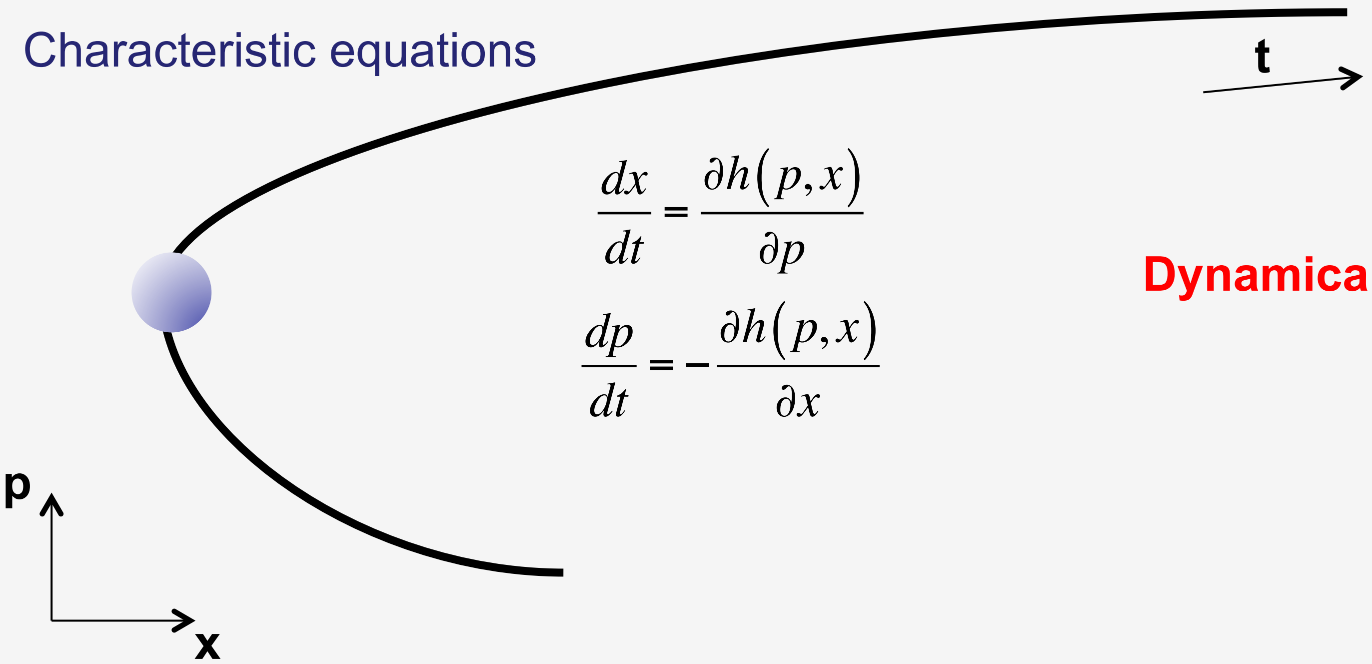
Quasi-particle trajectory

$$W_{t+\Delta t}\left(p,x\right)=W_t\left(p+\Delta t\frac{\partial h}{\partial x},x-\Delta t\frac{\partial h}{\partial p}\right)$$

Characteristic equations

$$\frac{dx}{dt}=\frac{\partial h\left(p,x\right)}{\partial p}$$

$$\frac{dp}{dt}=-\frac{\partial h\left(p,x\right)}{\partial x}$$



Dynamical map M

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Quasi-particle trajectory

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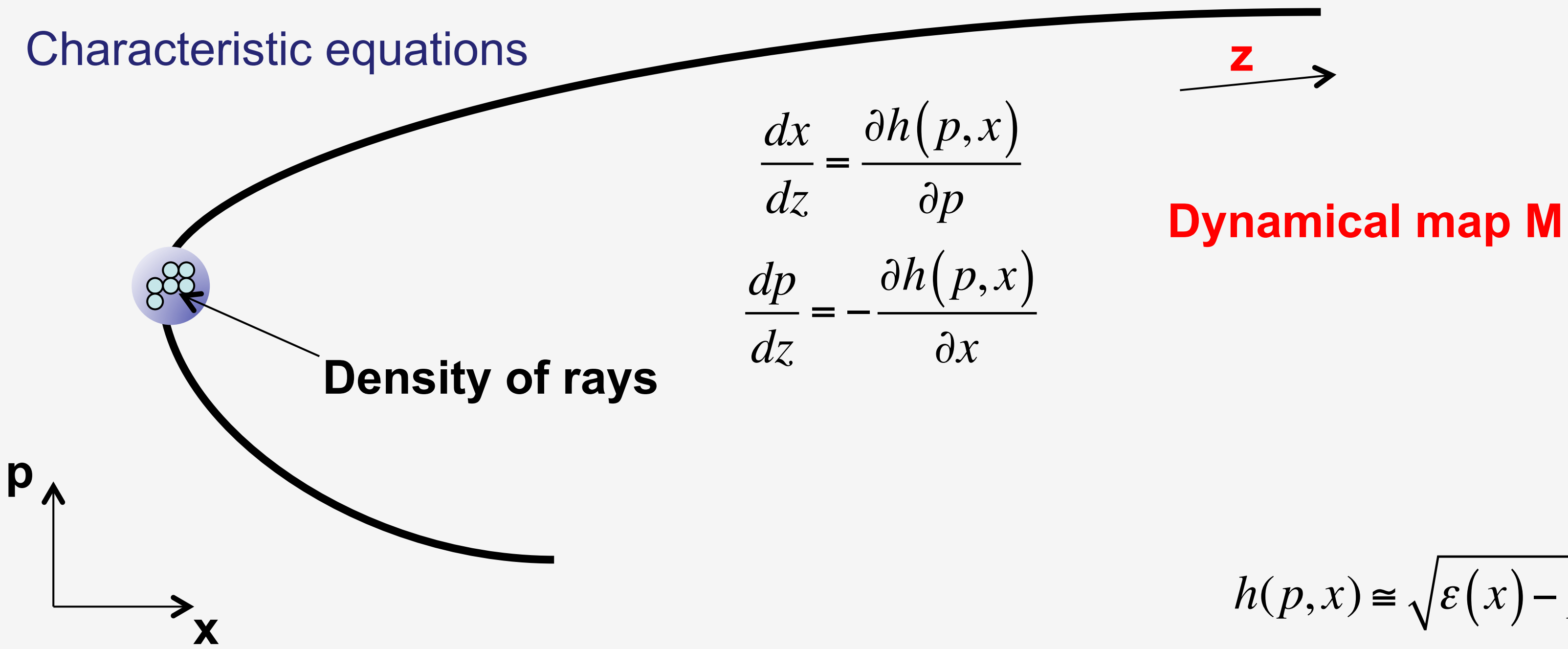
z-dependent problem $t \to -z$

$$W_{z+\Delta z}(p,x) = W_z\left(p + \Delta z \frac{\partial h}{\partial x}, x - \Delta z \frac{\partial h}{\partial p}\right)$$

Characteristic equations

$$\frac{dx}{dz} = \frac{\partial h(p,x)}{\partial p}$$

$$\frac{dp}{dz} = -\frac{\partial h(p,x)}{\partial x}$$



$$h(p,x) \cong \sqrt{\varepsilon(x) - p^2}$$

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Why and where relevant?

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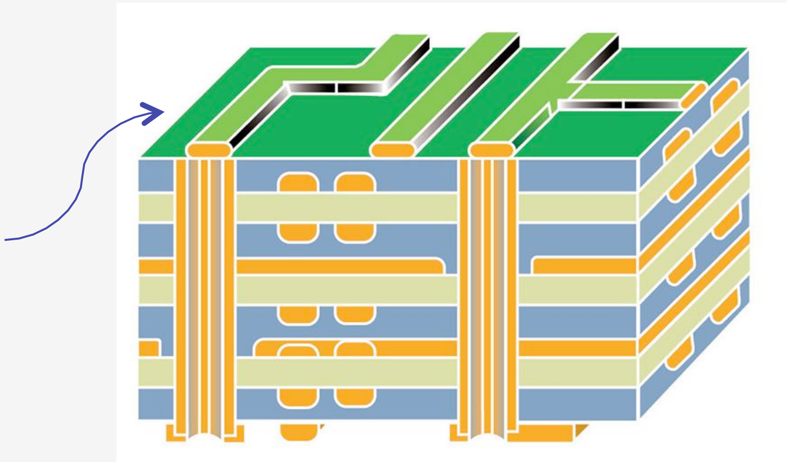
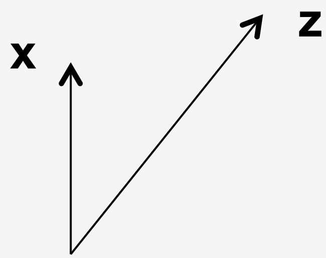
Inhomogeneous and charged medium, overmoded waveguides and cavities



$$k \equiv k(x)$$



Material dependence

Linearized transport equation*

$$\frac{\partial W_z(x,p)}{\partial z} \approx -\frac{\sqrt{1-p^2}}{p}\frac{\partial W_z(x,p)}{\partial x} - \frac{1}{kp}\frac{\partial k}{\partial x}\frac{\partial W_z(x,p)}{\partial p}$$

First-order PDE, can be solved by method of characteristics.

NOTE: higher order p- and x- derivatives for strong inhomogenities

*Besieris & Tappert;

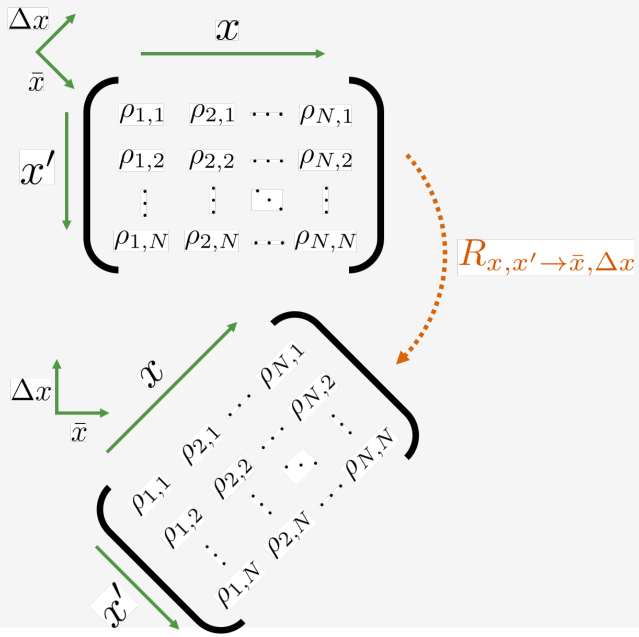
Wigner function in phase-space

- Ray/wave dynamical (position-momentum) space
- Intermediate representation in higher dimensional space
- **Direct relation with correlation function**

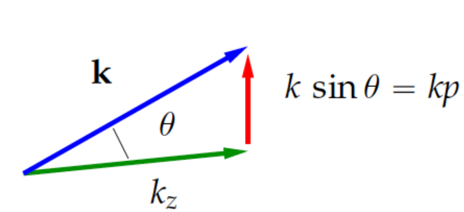
$$\bar{x} = \frac{x + x'}{2}$$

$$\Delta x = x - x'$$

$$W_z(x,p) = \left(\frac{k}{2\pi}\right)^d \int_{-\infty}^{\infty} \varphi_z\left(x + \frac{\Delta x}{2}\right)\varphi_z^*\left(x - \frac{\Delta x}{2}\right)\exp(-ikp\Delta x)\,ds$$



$$\varphi_z\left(x + \frac{\Delta x}{2}\right)\varphi_z^*\left(x - \frac{\Delta x}{2}\right) = \int_{-\infty}^{\infty} W_z(x,p;k)\exp(ikp\Delta x)\,dp$$



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Wigner function: properties

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- Realness
- Nonlinearity
- Non-positivity (smoothing needed) with positive projections

$$\int W(x,p) \quad dx = |\tilde{\varphi}(p)|^2$$

$$\int W(x,p) \quad dp = |\varphi(x)|^2$$

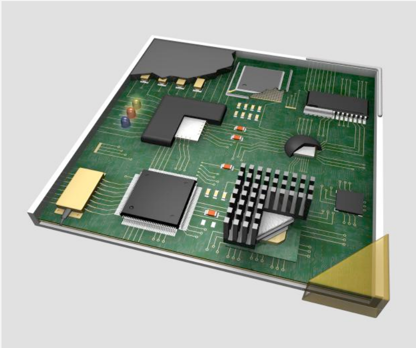
- Preserves signal normalization and range limitation

$$|W(x,p)| \leq \frac{\varphi\varphi^*}{\pi}$$

- Scale invariance

$$W(x,p) = W\left(\frac{x}{a}, ap\right)$$

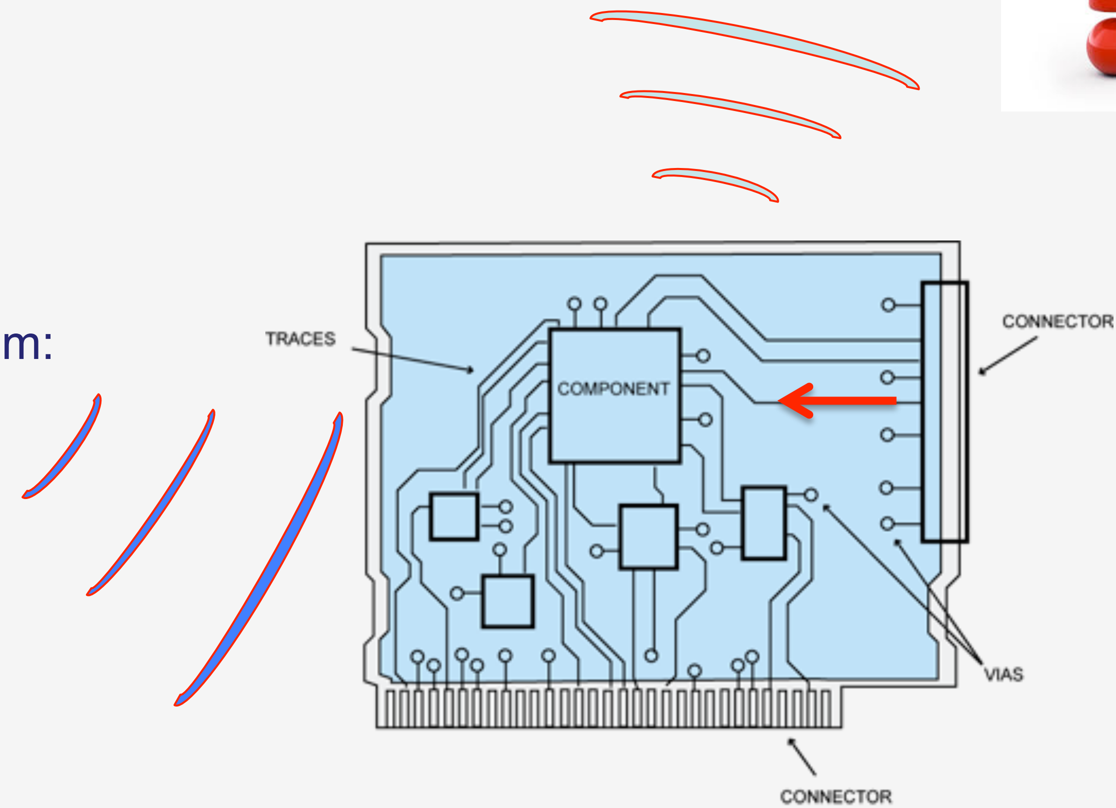
Motivation



Printed Circuit Board (PCB)

EMISSION problem: radiation from complex multifunctional electronics in both near- and far-field

SUSCEPTIBILITY problem:
solvable by reciprocity



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