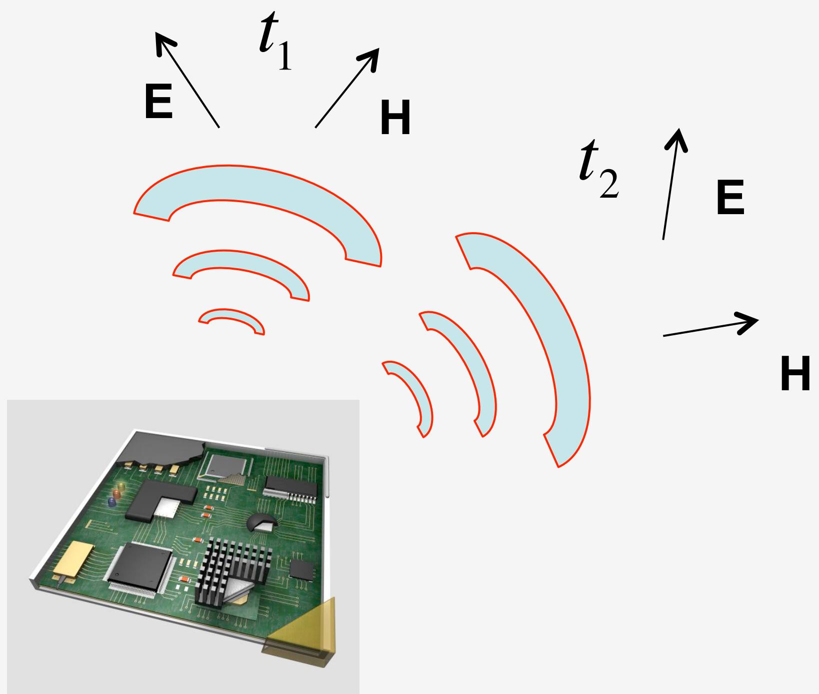
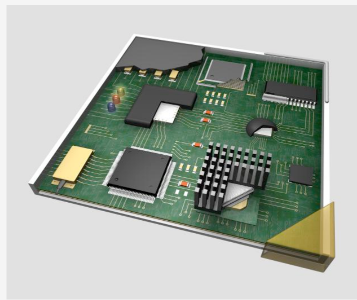
Global
Top 100
University

Motivation

The University of Nottingham
UNITED KINGDOM · CHINA · MALAYSIA





$$t_N = ?$$

Stochastic radiation pattern

$$dP_t\left(r,\Omega\right) = ?$$

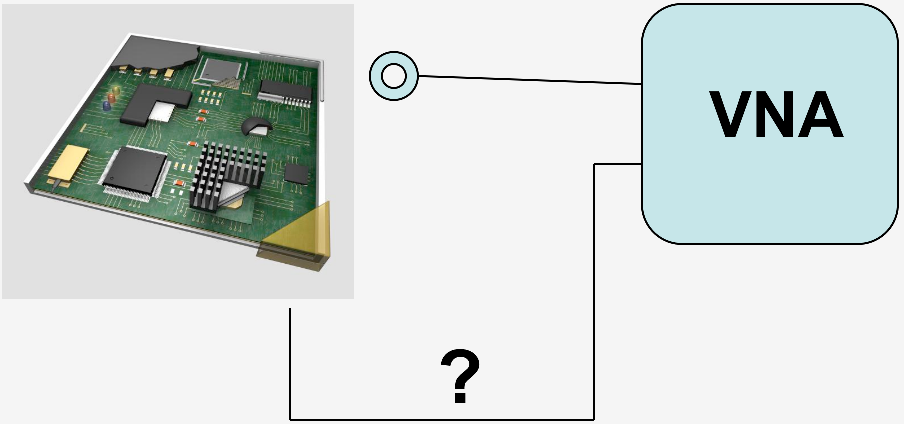
- Radiation pattern changes in frequency *and* time
- Changes occur rapidly
- Uncertain/unknown frequency *and* time behavior
- Random fields radiated in near- and far-field

Global
Top 100
University

General boundary conditions

The University of
Nottingham
UNITED KINGDOM · CHINA · MALAYSIA

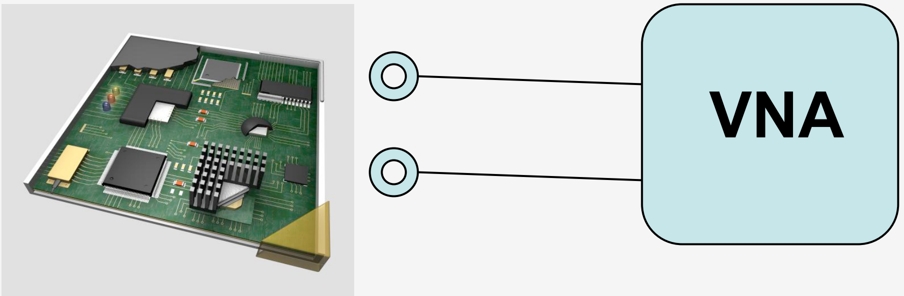
Idea: near-field measurement to predict far field propagation [Yaghijan]



- Scan simultaneously over large area, or…
- Scan sequentially with a phase-reference
- No phase-reference (trigger) in frequency (time)

Stochastic boundary conditions?

Idea: field-field correlation function [Bolomey]

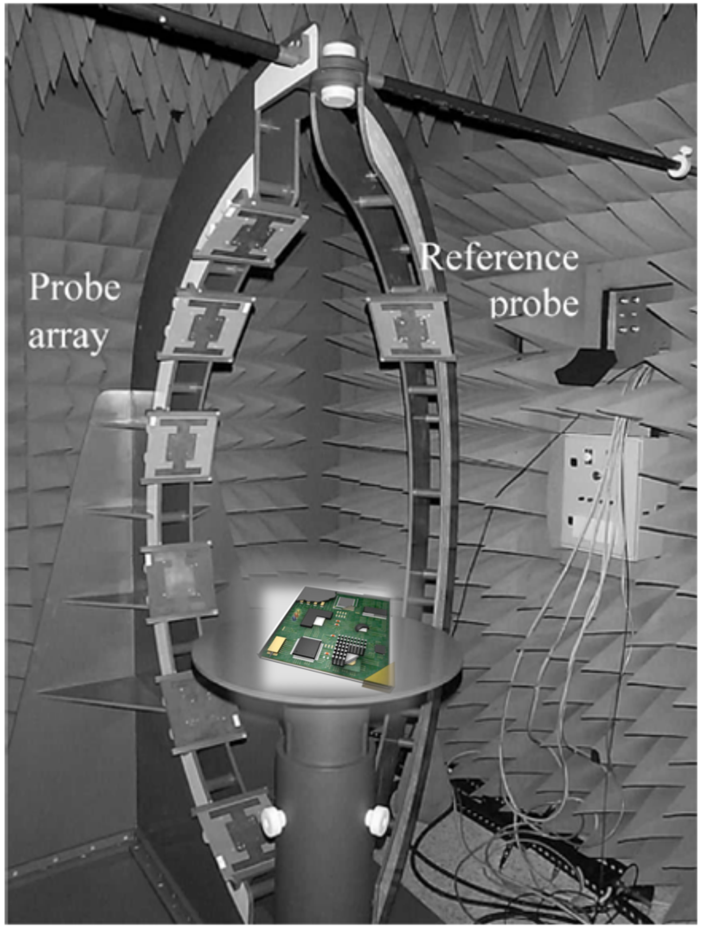


$$\tau = \tau_1, \tau_2, \ldots, \tau_N$$

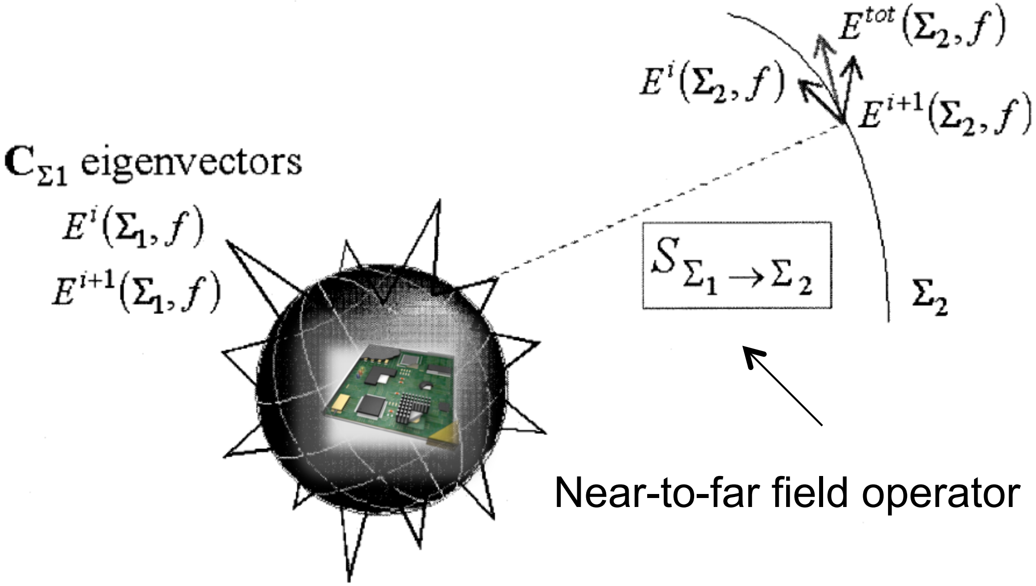
- Use additional probe as ``reference''
- *Near-field* dual probe system required
- Repeat measurements of field pairs

$$\Gamma_z(x_1, x_2) = \left\langle \varphi_z(x_1, \tau)\varphi_z^*(x_2, \tau) \right\rangle_\tau = \left\langle \rho_z(x_1, x_2; \tau) \right\rangle_\tau$$

Bolomey ring

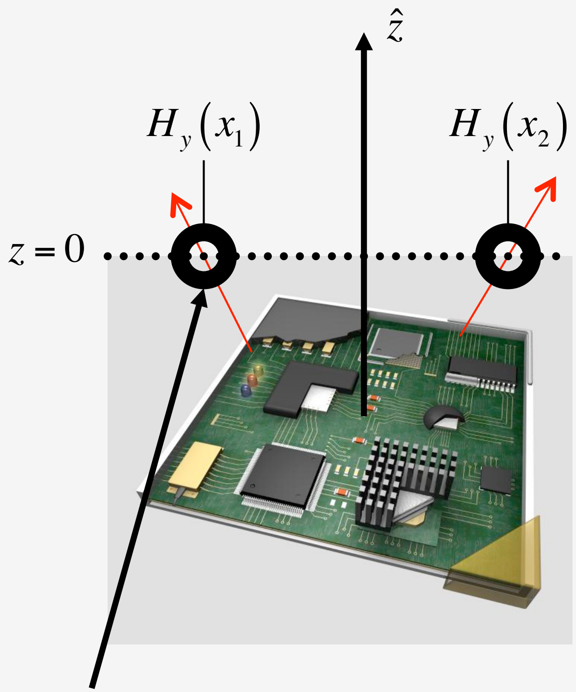


$$\Gamma_z(x_1,x_2) \leftrightarrow C_{\Sigma 1}$$



Physical problem

Complex wave source radiating in either free space or inside a cavity



Reference probe

$H_y(x,t)$= magnetic field

Propagation from *complex* wave sources

- Spatially extended
- Fast and random voltage/current signals
- Broad frequency band
- Non-stationary radiation

In absence of *phase reference*

$$\Gamma_{z=0}(x_1,x_2;\tau)=\left\langle H_y(x_1,t_1)H_y^*(x_2,t_2)\right\rangle_{\tau=t_1-t_2}$$

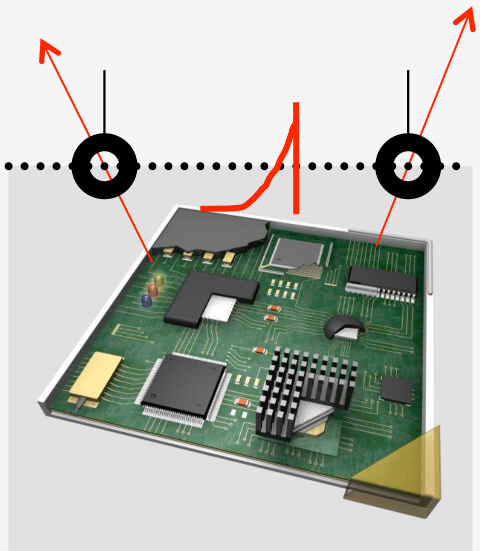
Given the correlation function near the source (*boundary conditions*), what is the correlation far from the source?

Correlation function in ``momentum'' space

Represent field on a Fourier (``momentum'') basis

$$\tilde{\Gamma}_z(p_1,p_2;\tau)=\left\langle \tilde{H}_y(p_1,t_1)\tilde{H}_y^*(p_2,t_2)\right\rangle_{\tau=t_1-t_2}$$

Use Huygens principle with *measured* boundary conditions



$$\tilde{\Gamma}_z(p_1,p_2)=\exp\left[ikz\left(h(p_1)-h^*(p_2)\right)\right]\tilde{\Gamma}_{z=0}(p_1,p_2)$$

$$h(p)=\begin{cases}\sqrt{1-p^2}, & p<1\\ i\sqrt{p^2-1}, & p>1\end{cases}$$

From probe *measured* fields near the source

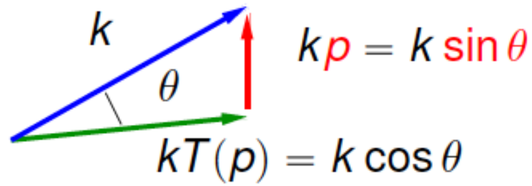
R. Littlejohn, R. Winston; J. Opt. Soc. Am. A **10** 9, 1993

Proof steps

- Use Green's second identity on source surface ∂S

$$\psi = \int_{\partial S} \left(G_0 \frac{\partial \psi}{\partial n} - \frac{\partial G_0}{\partial n} \psi \right) ds$$

- Change to momentum basis (partial Fourier transform) $\tilde{\psi}_z\,(p, t)$



- Calculate Dirichlet-to-Neumann condition

$$\frac{\partial \tilde{\psi}_0}{\partial n} = -ikT\,(p)\,\tilde{\psi}_0$$

- Arrive at propagated field $\tilde{\psi}_z\,(p, t) = e^{ikzT(p)}\,\tilde{\psi}_0\,(p, t)$, and correlation

$$T(p) = \begin{cases} \sqrt{1 - p^2}, & p < 1 \\ i\sqrt{p^2 - 1}, & p > 1 \end{cases}$$

$$\psi \rightarrow H_y$$
$$T \rightarrow h$$

Global
Top 100
University

Transport of Wigner functions

The University of Nottingham
UNITED KINGDOM · CHINA · MALAYSIA

$$W_z(x,p) = \left(\frac{k}{2\pi}\right)^d \int_{-\infty}^{\infty} \tilde{\Gamma}_z\left(p+\frac{q}{2}, p-\frac{q}{2}\right)\exp(-ikpq)\,dq$$

$$\uparrow$$

$$\tilde{\Gamma}_z(p_1,p_2) = \exp\left[ikz\left(T(p_1) - T^*(p_2)\right)\right]\tilde{\Gamma}_{z=0}(p_1,p_2)$$

$$\uparrow$$

$$\Gamma_{z=0}\left(p+\frac{q}{2}, p-\frac{q}{2}\right) = \int_{-\infty}^{\infty} W_{z=0}(x,p)\exp(ikqx)\,dx$$

Cascaded substitution yields an **exact transfer operator**

$$\boxed{W_z(x,p) = \int_{-\infty}^{\infty} G_z(x,x';p)W_{z=0}(x',p)\,dx'}$$

G. Gradoni, S. C. Creagh, G. Tanner, C. Smartt, D. W. P. Thomas, “A Phase-Space Approach for Propagating Field-Field Correlation Functions,” New J. Phys. **17** (2015) 093027

Exact transfer operator

$$G_z\left(x,x';p,p'\right)=\delta\left(p-p'\right)\int_{-\infty}^{\infty} e^{ik\left(x-x'\right)q+ikz\left[h\left(p+\frac{q}{2}\right)-h^*\left(p-\frac{q}{2}\right)\right]}dq$$

1. **Propagating waves** $$G_z\left(x,x';p,p'\right)\cong\delta\left(p-p'\right)\delta\left(x-x'-z\frac{p}{\sqrt{1-p^2}}\right)$$

2. **Evanescent waves** $$G_z\left(x,x';p,p'\right)\cong\delta\left(p-p'\right)e^{-2kz\sqrt{p^2-1}}$$

G. Gradoni, S. C. Creagh, G. Tanner, C. Smartt, D. W. P. Thomas, “A Phase-Space Approach for Propagating Field-Field Correlation Functions,” New J. Phys. **17** (2015) 093027

Transport of Wigner functions

At leading order we find

Map M^{-1} of the classical trajectories

$$G_z\left(x,x';p\right)\approx\begin{cases}\delta\left(x'-x+z\dfrac{p}{\sqrt{1-p^2}}\right), & p<1\\ \delta\left(x'-x\right)e^{i2kz\sqrt{p^2-1}}, & p>1\end{cases}$$

thus

$$W_z\left(x,p\right)\approx\begin{cases}W_{z=0}\left(x-z\dfrac{p}{\sqrt{1-p^2}},p\right), & p<1\\ W_{z=0}\left(x,p\right)e^{-2kz\sqrt{|p|^2-1}}, & p>1\end{cases}$$

Perron-Frobenius operator ($p<1$)

Evanescent operator ($p>1$)

The leading order approximation is:

$$W_z\left(x,p;k\right)=W_{z=0}\left(x-z\frac{p}{\sqrt{1-p^2}},p;k\right)$$

Global Top 100 University

Wigner and correlation functions

The University of Nottingham
UNITED KINGDOM · CHINA · MALAYSIA

In presence of stochastic field, incomplete field information: ensemble average

$$\Gamma_{z=0}\left(x_1,x_2\right)=\left\langle\varphi_z\left(x_1,t_1;\tau\right)\varphi_z^*\left(x_2,t_2;\tau\right)\right\rangle_\tau$$

$$W_z\left(x,p;k\right)=\left(\frac{k}{2\pi}\right)^d\int_{-\infty}^{\infty}\Gamma_z\left(x+\frac{s}{2},x-\frac{s}{2};k\right)\exp\left(-ikps\right)ds$$

Wigner transport equation

$$\Gamma_z\left(x+\frac{s}{2},x-\frac{s}{2};k\right)=FT\left\{W_{z=0}\left(x-z\frac{p}{\sqrt{1-p^2}},p;k\right)\right\}$$

Model
Correlation functions

Spatial coherence as a model for the random emission of complex sources

Partially coherent (Schell model)

Coherence factor:

$$\Gamma(x_1,x_2)=\mathrm{I}(x_1,x_2)\mu(x_1-x_2)$$

$$0\le\mu(x_1-x_2)\le 1$$

Quasi-homogeneous

$$\Gamma(x_1,x_2)=\mathrm{I}\left(\underbrace{\frac{x_1+x_2}{2}}_{\mathbf{x}}\right)\mu(\underbrace{x_1-x_2}_{\mathbf{s}})$$

Example:
Gauss-Schell **in statistical optics**

Uncorrelated/incoherent

$$\Gamma(x_1,x_2)=\mathrm{I}\left(\frac{x_1+x_2}{2}\right)\delta(x_1-x_2)$$

Completely coherent

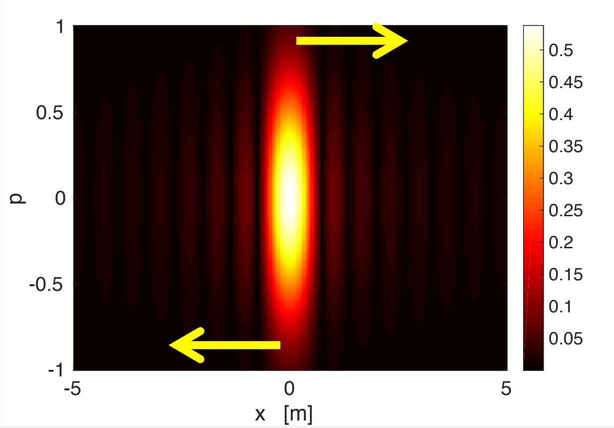
$$\Gamma(x_1,x_2)=f_1(x_1)f_2(x_2)$$

Global
Top 100
University

Gaussian Schell model: propagating waves

The University of Nottingham
UNITED KINGDOM · CHINA · MALAYSIA

Source correlation



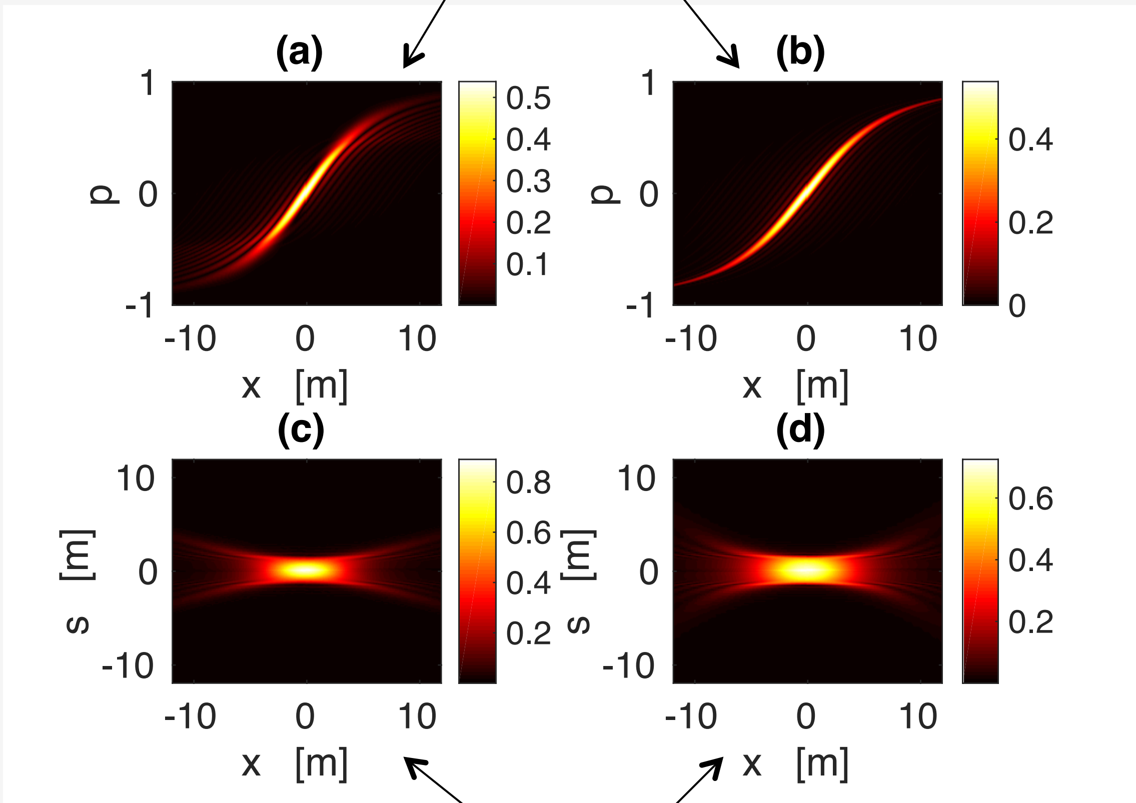
$$\lambda = 0.3m$$
$$\sigma_x = 1.0m$$
$$\sigma_s = 0.1m$$

$$W_{z=0}(x,p) = \sqrt{2\pi}\sigma_s I_0 \exp\left[-\frac{x^2}{2\sigma_x^2} - \frac{k^2 p^2 \sigma_s^2}{2}\right]$$

$$\boxed{\frac{p}{\sqrt{1-p^2}}}$$

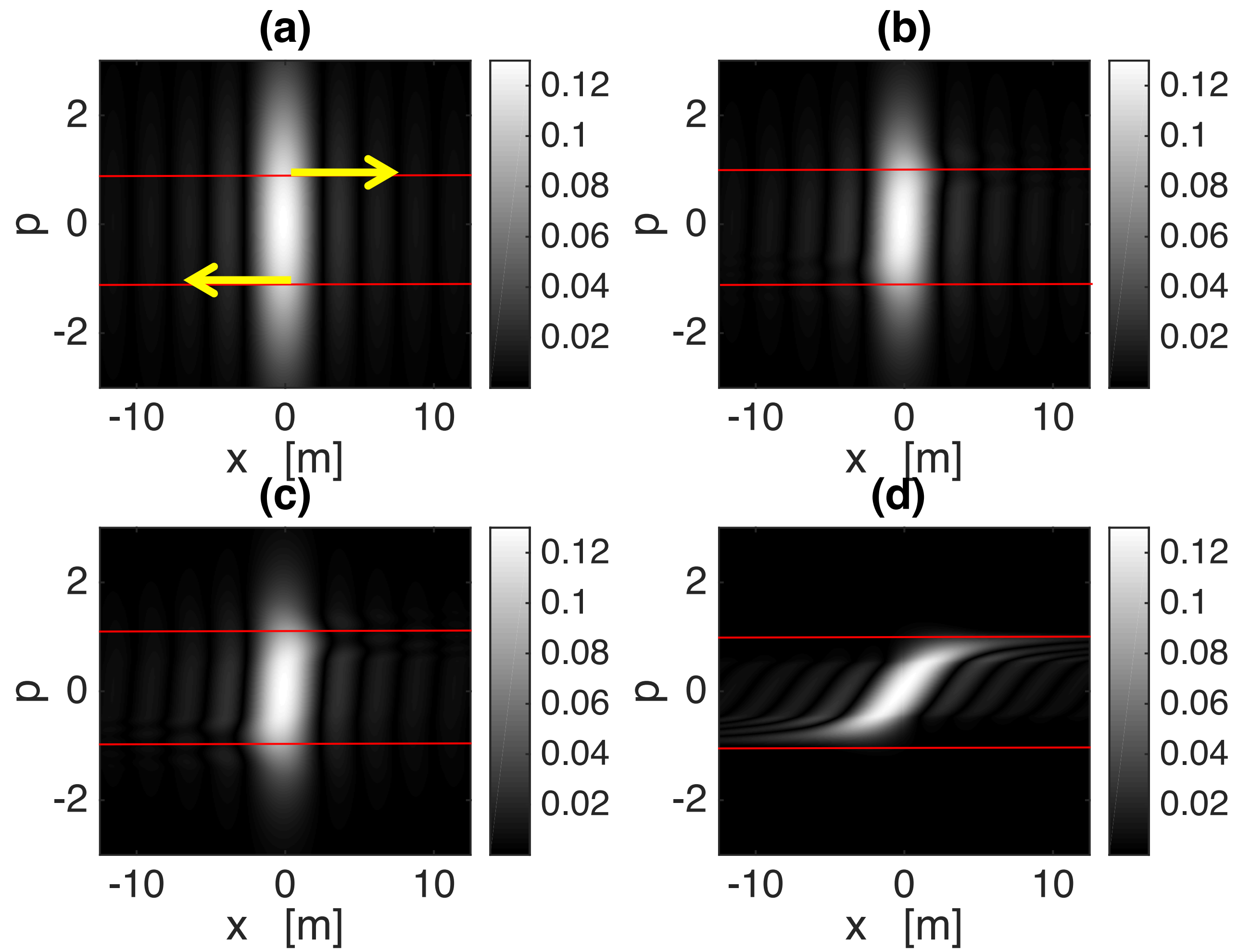
Exact **Perron-Frobenius**

$$W_{z=10\lambda}(x,p)$$



$$\Gamma_{z=10\lambda}(x,s)$$

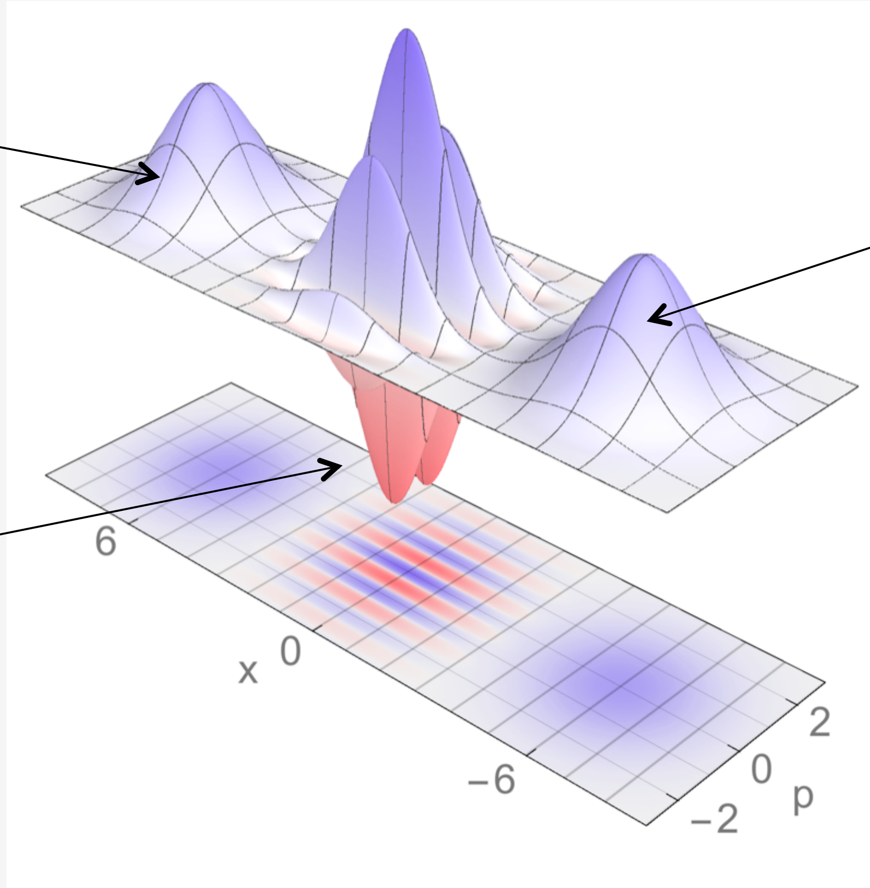
Gaussian Schell model: evanescent waves



Effect of interference on Wigner's functions

Gaussian beam

Gaussian beam



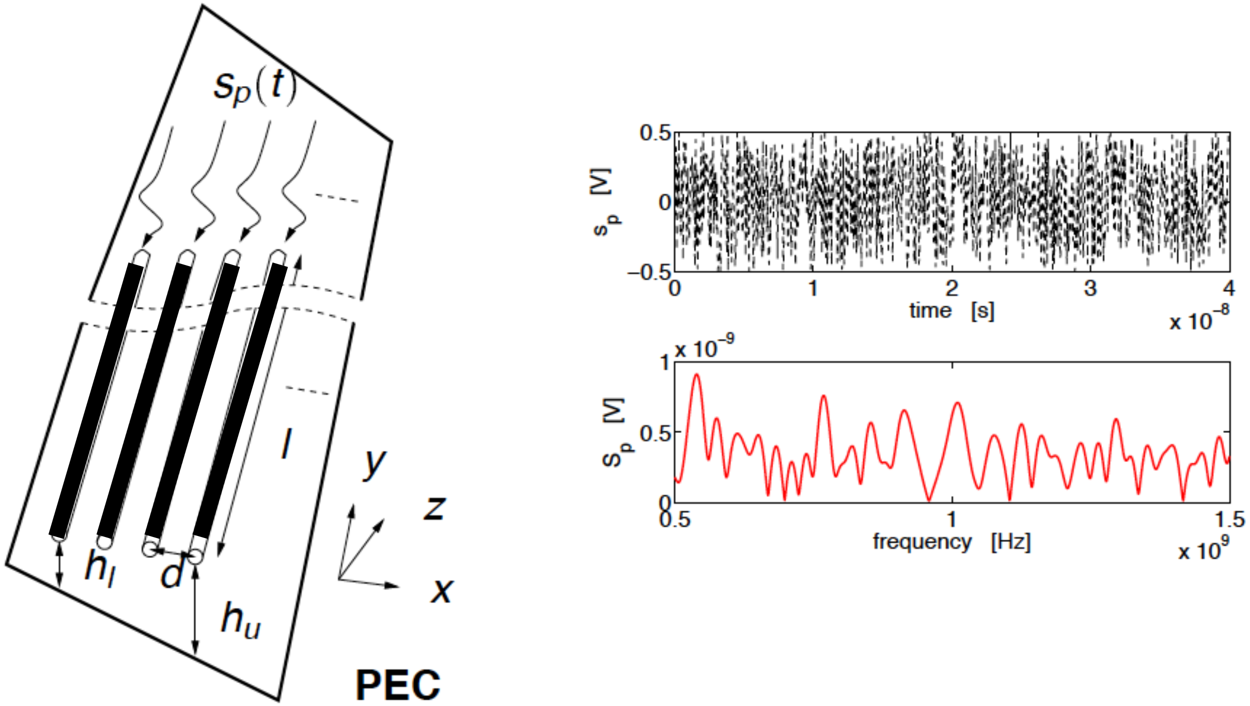
Fringes from interference: additional state

Global
Top 100
University

Realistic statistical source

The University of Nottingham
UNITED KINGDOM · CHINA · MALAYSIA

Realistic scenario: wire array on a ground plane



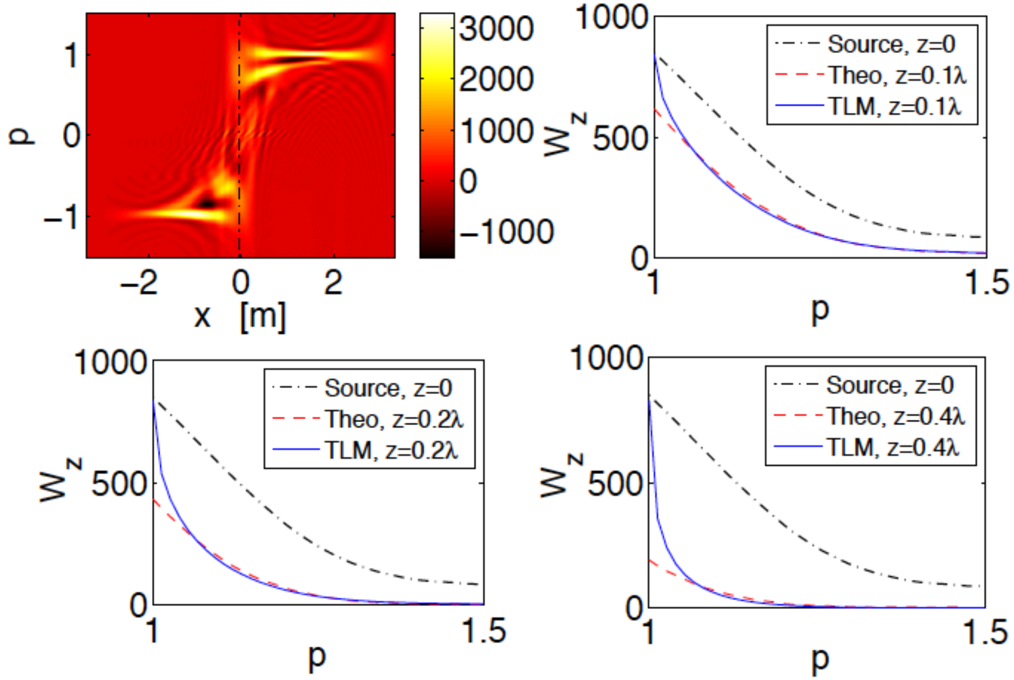
Random (uniformly distributed) voltages driving cable wires

Global
Top 100
University

Full wave TLM simulations

The University of Nottingham
UNITED KINGDOM · CHINA · MALAYSIA

Near-field correlation: evanescent waves



Credit: Christopher Smartt & David Thomas, George Green Institute for Electromagnetic Research, UoN